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THE PROPAGATION OF INITIALLY PLANE WAVES IN NONHOMOGENEOUS VISCOELASTIC MEDIA

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Abstract—In this study, the propagation of an initially plane wave in a linear isotropic nonhomogeneous viscoelastic medium, where the nonhomogeneity varies transversely to the direction of propagation, is investigated. For this purpose, first the propagation of waves in a linear isotropic viscoelastic medium of arbitrary inhomogeneity is studied by employing the notion of singular surfaces. The characteristic equation governing wave velocities, and the growth and decay equations describing the change of the strength of the discontinuity as the wave front moves are obtained.

In the second part of this work, the propagation of initially plane waves is studied for three types of inhomogeneities by employing the findings established in the first part. The first kind of inhomogeneity considered is of axisymmetrical type where the wave propagation velocity depends on the radial coordinate only, increasing linearly up to a certain radial distance and remaining constant thereafter. The second kind is also axisymmetrical with a wave velocity distribution decreasing linearly till a given value of the radial coordinate. In the third one, the wave velocity is assumed to vary linearly over a given interval along a certain coordinate axis only, which is perpendicular to the direction of propagation, and remain constant outside. The ray and wave front analyses are carried out and the decay or growth of stress and velocity discontinuities are studied for each of the three cases.

1. INTRODUCTION

Study of propagation of waves in nonhomogeneous elastic and viscoelastic media is of great importance in understanding the dynamical response of both composite materials an materials with local impurities. However, due to mathematical complexities involved, only some problems where the inhomogeneities are assumed to depend on the coordinate coinciding with the direction of propagation have been investigated (see, e.g. [1–4]). An extensive list of references dealing with nonhomogeneities of this type may be found in [4]. In Refs. [1–3], the theory of propagating surfaces of discontinuities is employed and the solutions for stress and particle velocity are expressed as Taylor series expansions about the time of arrival of the wave front. In Ref. [4], Longcope and Steele developed approximate solutions for pulse propagation in nonhomogeneous elastic media where the ratio of the pulse length to a characteristic length of the wave speed variation is assumed to be small. Karal and Keller [5], and Steele [6] proposed general wave front analyses for nonhomogeneous elastic media. Steele [6] also gave an interesting example where the nonhomogeneity depends on the direction transverse to that of wave propagation. Apart from these studies the work by Ting and Lee [7] should also be cited where they investigated the propagation of an initially plane wave front in a linear elastic composite medium containing cylindrical or spherical reinforcing elements, which can, in a sense, be interpreted as the inclusion of the inhomogeneity transverse to the propagation direction.

In this study, we investigate the propagation of an initially plane wave in a linear isotropic nonhomogeneous viscoelastic medium where the nonhomogeneity varies transversely to the direction of propagation. To this end, we first study the propagation of waves in a linear isotropic viscoelastic medium of arbitrary inhomogeneity by employing the notion of singular surfaces and by extending the analysis, presented by Valanis [8] for the homogeneous case. The characteristic equation, obtained through the use of the compatibility equations discussed in detail by Thomas [9], reveals the existence of the well-known longitudinal and transverse waves whose velocities depend on the spatial coordinates due to inhomogeneity. The growth and decay equations, describing the change of the strength of the discontinuity as the wave front moves, are then derived for longitudinal and transverse waves, separately. These equations indicate that there are three factors which can possibly influence the decay and growth. They represent the effects of inhomogeneity, geometry of the wave front and material internal friction, respectively. It is further observed that the growth and decay due to geometry and internal friction are indirectly affected by the inhomogeneity. Since the decay and growth equations are valid along the rays, they can only be integrated explicitly if the equations of rays are known. The rays are governed by a set of first order ordinary differential equations discussed in detail by Jeffrey and Taniuti[10] for a nonlinear anisotropic medium, which for the sake of completeness are presented in the reduced form for our linear isotropic inhomogeneous material.

In the second part of this work, we investigate the propagation of initially plane waves for three types of inhomogeneities by employing the findings established in the first part. The first kind of inhomogeneity considered is of axisymmetrical type where the wave propagation velocity depends on the radial coordinate only, increasing linearly up to a certain radial distance and remaining constant thereafter (Fig. 2). The second kind is also axisymmetrical with a wave velocity distribution decreasing linearly till a given value of the radial coordinate and remaining constant beyond (Fig. 3). In the third kind, the wave velocity is assumed to vary linearly over a given interval along a certain coordinate axis only, which is perpendicular to the direction of propagation, and remain constant outside (Fig. 4).

The analysis indicates that the rays in all these three cases are circular arcs in the linear region whereas they are straight lines outside (Figs. 2–4). The wave fronts, obtained by forming the loci of the points on the rays at a fixed time, are also shown in the figures. They are composed of two

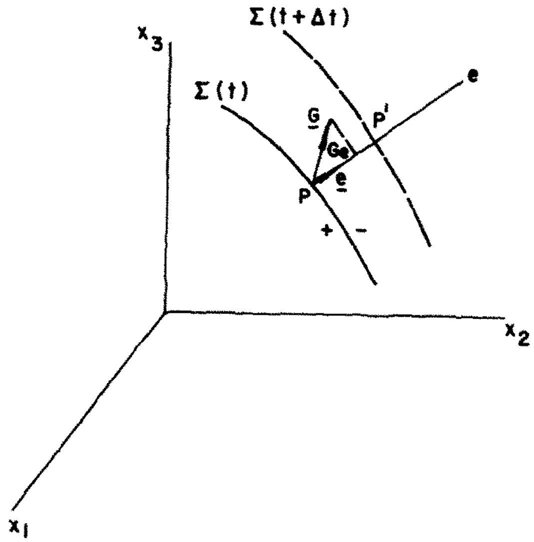

Fig. 1. Geometrical description of a surface of discontinuity.

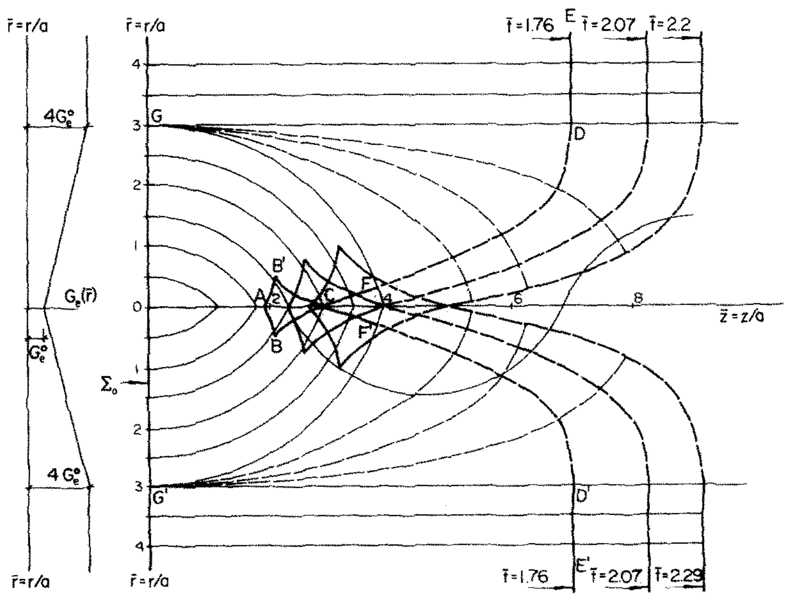

Fig. 2. Rays and wave fronts for axisymmetrical inhomogeneity with radially increasing wave velocity.

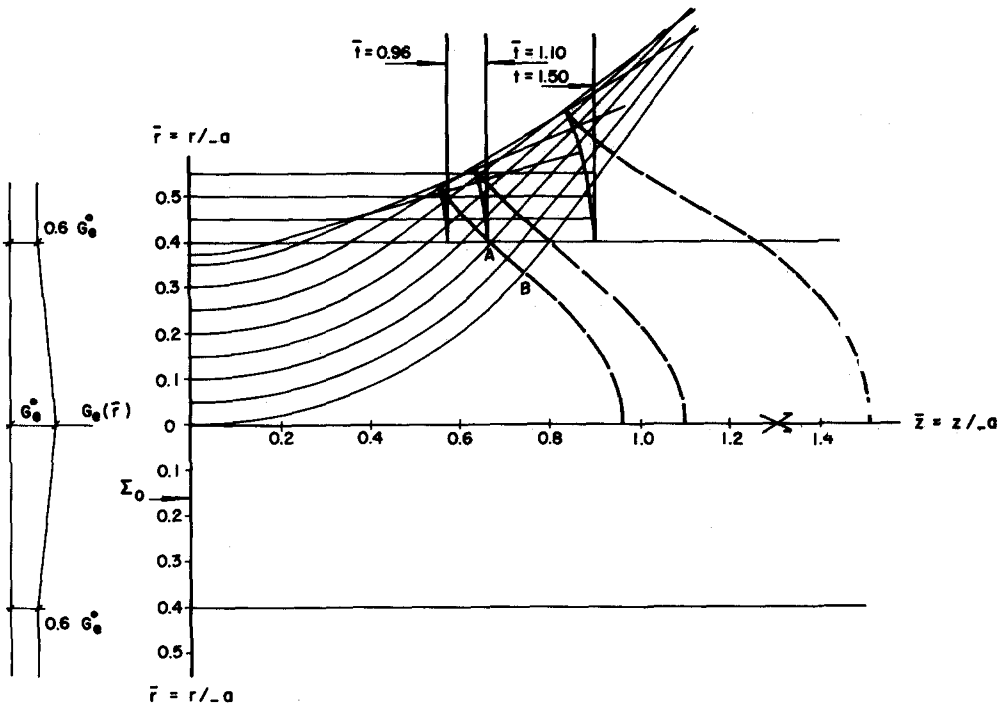

Fig. 3. Rays and wave fronts for axisymmetrical inhomogeneity with radially decreasing wave velocity.

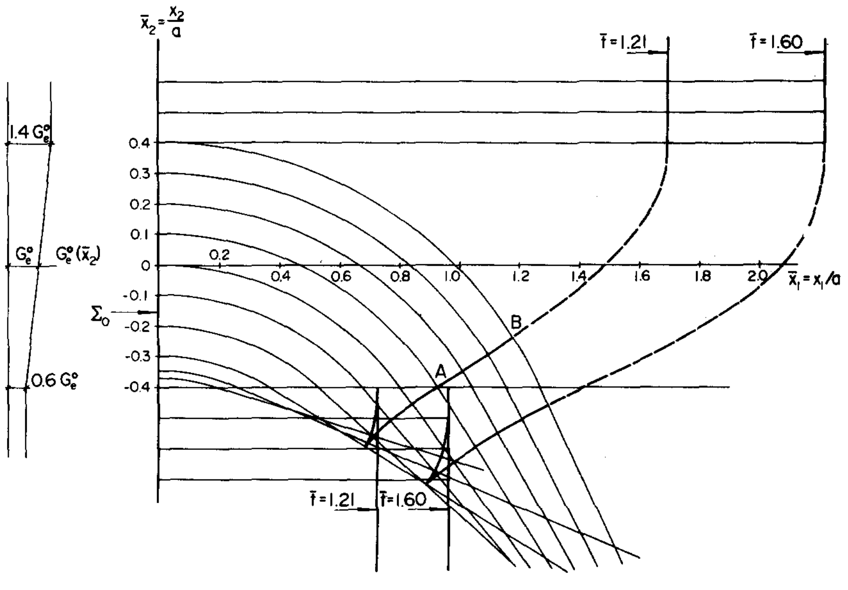

Fig. 4. Rays and wave fronts for unidirectional inhomogeneity.

parts, which we call leading and lagging wave fronts, bringing initial and additional disturbances to undisturbed and already disturbed material points, respectively. In the figures the secondary rays and wave fronts caused by the discontinuities in the derivatives of the wave velocity distributions are shown by dotted lines. The growth and decay equations along the circular rays before they intersect the axis of symmetry in the first case and before they enter the homogeneous region in the other two cases are also obtained. These equations exhibit a decay due to material internal friction, and the stress decay and velocity growth due to material inhomogeneity in all the three cases. The geometrical growth or decay caused by the inhomogeneity appears as a growth in the first, decay in the second and disappears in the third problem. The analysis further reveals that the axis of symmetry in the first problem is a singular line along which the velocity and stress grows beyond limit, and that, in the other two problems caustics can possibly form in the homogeneous region due to the converging rays initiating from the linear region.

2. EQUATIONS OF GEOMETRICAL, KINEMATICAL AND DYNAMICAL COMPATIBILITY

In the present study we investigate the propagation of waves in nonhomogeneous linear isotropic viscoelastic media. As the resulting disturbance moves through the medium, the behavior is best understood if we use the notion of wave fronts. A wave front is defined as the boundary between the disturbed and undisturbed regions or the initially disturbed region and the one having additional disturbance. When the material at a point becomes suddenly disturbed from an undisturbed state or when an already disturbed material point undergoes some additional disturbance, it can only be so if the dependent variables, such as stresses, velocities, etc. or some of their derivatives suffer finite jumps at the point. In our study we will consequently treat the wave front as a surface of discontinuity and show its configuration at the time t by $\Sigma(t)$ (see Fig. 1). We refer the surface $\Sigma(t)$to a cartesian coordinate system (x_1, x_2, x_3) and denote its unit normal vector, in the moving direction of $\Sigma(t)$, by $\mathbf{e}$. We show the moving side of $\Sigma(t)$ by $(-)$ and the opposite side by $(+)$. If Z describes a physical quantity, which depends on both the position $\mathbf{x}$ and the time t, its finite jump across $\Sigma(t)$ will be designated by a square bracket, defined by

$$[Z] = Z^+ - Z^-. \tag{2.1}$$

The parametric equation of the surface $\Sigma(t)$ has the form

$$x_i = x_i(u^\alpha, t), \tag{2.2}$$

where u^α denotes the surface coordinates of $\Sigma(t)$ and the superscript α takes the values 1 and 2. In the succeeding discussions the Greek indices are assumed to take the values 1 and 2, whereas the Latin indices range from 1 to 3.

There are three types of compatibility equations; namely, geometrical, kinematical and dynamical equations, which are to be satisfied on the surfaces of discontinuity. These compatibility conditions are investigated in detail by Thomas[9]. For the sake of completeness we here review them very briefly.

We first define, on $\Sigma(t)$, the discontinuity of the quantity Z and that of its normal derivatives with respect to the normal coordinate e (see Fig. 1) by

$$[Z] = A; \quad \left[\frac{\partial Z}{\partial e}\right] = B; \quad \left[\frac{\partial^2 Z}{\partial e^2}\right] = C. \tag{2.3}$$

The equations of geometrical compatibility, relating the partial derivatives of Z with respect to x_i to A, B and C, then become

$$\left[\frac{\partial Z}{\partial x_i}\right] = Be_i + g^{\alpha\beta}A_{,\alpha}x_{i\beta}$$
$$\left[\frac{\partial^2 Z}{\partial x_i \partial x_j}\right] = Ce_ie_j + g^{\alpha\beta}(B_{,\alpha} + g^{\sigma\tau}b_{\alpha\sigma}A_{,\tau})(e_ix_{j\beta} + e_jx_{i\beta}) + g^{\alpha\beta}g^{\sigma\tau}(A_{,\alpha\sigma} - Bb_{\alpha\sigma})x_{i\beta}x_{j\tau}, \tag{2.4}$$

where $g^{\alpha\beta}$ = contravariant components of the first fundamental tensor of $\Sigma(t)$; $b_{\alpha\sigma}$ = covariant components of the second fundamental tensor of $\Sigma(t)$; $x_{i\beta} = (\partial x_i/\partial u^\beta)$ and the subscript after comma designates differentiation, e.g. $A_{,\alpha} = \partial A/\partial u^\alpha$. In eqns (2.4) and the subsequent discussions the indicial notation and the rules pertaining to its use are employed.

The equations of kinematical compatibility, relating the partial derivatives of Z with respect to time t to A, B and C, are

$$\left[\frac{\partial Z}{\partial t}\right] = -BG_e + \frac{\delta A}{\delta t}$$
$$\left[\frac{\partial^2 Z}{\partial t^2}\right] = CG_e^2 - 2G_e\frac{\delta B}{\delta t} - B\frac{\delta G_e}{\delta t} - G_e g^{\alpha\beta}A_{,\alpha}G_{e,\beta} + \frac{\delta^2 A}{\delta t^2}, \tag{2.5}$$

where G_e is the normal component of the propagation velocity of $\Sigma(t)$ and the δ-time derivative of a function $f(x_i, t)$ is defined by

$$\frac{\delta f}{\delta t} = \lim_{\Delta t \to 0} \frac{f(P') - f(P)}{\Delta t}.$$

The points P and P' appearing in this expression are shown in Fig. 1.

The cartesian form of the dynamical equations of compatibility, related to the conservation of mass and linear momentum, are

$$\begin{aligned} &[\rho(G_e - v_e)] = 0 \\ &e_j[\sigma_{ji}] = -\rho(G_e - v_e)[v_i], \end{aligned} \tag{2.6}$$

respectively. In eqns (2.6), ρ is the mass density, v_i and v_e are the cartesian and normal components of the particle velocity and σ_{ij} denote the components of the stress tensor.

3. EQUATIONS OF DECAY AND GROWTH

In subsequent discussions the body is referred to a cartesian coordinate system (x_1, x_2, x_3). We first obtain the possible wave velocities by which the disturbances propagate in a nonhomogeneous linear isotropic viscoelastic medium. We start our analyses by assuming that the displacement **u** is continuous while the velocity and stress may suffer finite jumps on $\Sigma(t)$, i.e.

$$[u_i] = 0; \quad [v_i] \neq 0; \quad [\sigma_{ij}] \neq 0. \tag{3.1}$$

If v_e is assumed to be small compared to the normal wave velocity G_e, the first of eqns (2.6) implies that ρ is continuous on $\Sigma(t)$. The equation of dynamical compatibility of linear momentum, the second of eqns (2.6), then takes the form

$$e_j[\sigma_{ji}] = -\rho G_e[v_i]. \tag{3.2}$$

By writing the linear isotropic viscoelastic constitutive equations

$$\begin{aligned} \sigma_{ij}(t) = 2\mu(0)\epsilon_{ij}(t) + \delta_{ij}\lambda(0)\epsilon_{kk}(t) + \int_0^t 2\mu'(t-\tau)\epsilon_{ij}(\tau)\,d\tau \\ + \delta_{ij}\int_0^t \lambda'(t-\tau)\epsilon_{kk}(\tau)\,d\tau \end{aligned} \tag{3.3}$$

on both sides of $\Sigma(t)$ and taking their differences we get

$$[\sigma_{ij}] = 2\mu(0)[\epsilon_{ij}] + \delta_{ij}\lambda(0)[\epsilon_{kk}]. \tag{3.4}$$

In eqn (3.3), ϵ_{ij} denote the components of the infinitesimal strain tensor

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{3.5}$$

$\mu(t)$, $\lambda(t)$ are the Lame's relaxation functions of the viscoelastic material, δ_{ij} is the usual Kronecker delta and prime designates differentiation with respect to the time argument. In writing eqn (3.3), the space dependences of σ_{ij}, ϵ_{ij}, μ and λ are not indicated explicitly for the sake of brevity.

In view of the assumption $[u_i] = 0$, the geometrical and kinematical conditions, eqns (2.4) and (2.5), and the strain-displacement relations, eqn (3.5), we obtain

$$\begin{aligned} &[\epsilon_{ij}] = \frac{1}{2}(\alpha_i e_j + \alpha_j e_i) \\ &[v_i] = -G_e \alpha_i, \end{aligned} \tag{3.6}$$

where α_i stands for the discontinuity defined by

$$\alpha_i = \left[\frac{\partial u_i}{\partial e}\right]. \tag{3.7}$$

The linear momentum equation, eqn (3.2), with the aid of eqns (3.4) and (3.6), leads to the well-known characteristic equation

$$\{\mu(0) - \rho G_e^2\}\alpha_i + \{\mu(0) + \lambda(0)\}\alpha_k e_k e_i = 0, \tag{3.8}$$

revealing the existence of longitudinal and transverse waves with the propagation velocities

$$G_e^2 = \begin{cases} \dfrac{\lambda(0) + 2\mu(0)}{\rho} & \text{for longitudinal waves} \\ \\ \dfrac{\mu(0)}{\rho} & \text{for transverse waves} \end{cases} \tag{3.9}$$

and with the discontinuity vectors satisfying

$$\boldsymbol{\alpha} \parallel \mathbf{e} \quad \text{for longitudinal waves}$$
$$\boldsymbol{\alpha} \perp \mathbf{e} \quad \text{for transverse waves.}$$

We here note that the wave velocities depend on the spatial coordinates which result from the nonhomogeneity of the medium.

We now start studying the growth and decay equations, which describe the change of the strength of the discontinuity as the wave front moves. We begin our analyses with the equation of linaear momentum

$$\sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i, \tag{3.10}$$

where **f** is the body force and dot denotes differentiation with respect to time. When eqn (3.10) is written on both sides of $\Sigma(t)$ and their difference is taken we get

$$[\sigma_{ji,j}] = \rho[\ddot{u}_i], \tag{3.11}$$

provided that the body force **f** is continuous on $\Sigma(t)$. As a first step in evaluating the left hand side of eqn (3.11), we differentiate the constitutive equations, eqn (3.3), with respect x_j. We thus obtain

$$\begin{aligned} \sigma_{ji,j}(t) &= 2\mu_{,j}(0)\epsilon_{ji}(t) + 2\mu(0)\epsilon_{ji,j}(t) + \lambda_{,i}(0)\epsilon_{kk}(t) + \lambda(0)\epsilon_{kk,i}(t) \\ &+ \left\{\int_0^t 2\mu'(t-\tau)\epsilon_{ji}(\tau)\,\mathrm{d}\tau\right\}_{,j} + \left\{\int_0^t \lambda'(t-\tau)\epsilon_{kk}(\tau)\,\mathrm{d}\tau\right\}_{,i}. \end{aligned} \tag{3.12}$$

Since the integrals in brackets are continuous at the wave front, the first of the geometrical and the first of the kinematical compatibility equations, eqns (2.4) and (2.5), lead to

$$\begin{aligned} \left[\left\{\int_0^t 2\mu'(t-\tau)\epsilon_{ji}(\tau)\,\mathrm{d}\tau\right\}_{,j}\right] &= -\frac{2\mu'(0)}{G_e}[\epsilon_{ji}]e_j \\ \left[\left\{\int_0^t \lambda'(t-\tau)\epsilon_{kk}(\tau)\,\mathrm{d}\tau\right\}_{,i}\right] &= -\frac{\lambda'(0)}{G_e}[\epsilon_{kk}]e_i. \end{aligned} \tag{3.13}$$

When eqns (3.5), $(3.6)_1$ and (3.13) are used, eqn (3.12) yields

$$[\sigma_{ji,j}] = \mu_{,j}(0)(\alpha_i e_j + \alpha_j e_i) + \lambda_{,i}(0)\alpha_k e_k + (\mu(0) + \lambda(0))[u_{j,ji}] \\ + \mu(0)[u_{i,jj}] - \frac{\mu'(0) + \lambda'(0)}{G_e}\alpha_j e_j e_i - \frac{\mu'(0)}{G_e}\alpha_i. \tag{3.14}$$

With the aid of the second of the equations of geometrical compatibility, eqn $(2.4)_2$, the assumption that the displacement is continuous on $\Sigma(t)$, and the relations

$$x_{k\alpha}x_{k\beta} = g_{\alpha\beta}; \quad x_{k\alpha}e_k = 0, \tag{3.15}$$

we obtain

$$[u_{i,kk}] = \alpha_i^* - 2\Omega\alpha_i \\ [u_{k,ki}] = \alpha_k^* e_i e_k + g^{\alpha\beta}(\alpha_{k,\alpha})(e_i x_{k\beta} + e_k x_{i\beta}) - g^{\alpha\beta}g^{\sigma\tau}\alpha_k b_{\alpha\sigma}x_{i\beta}x_{\kappa\tau}, \tag{3.16}$$

where $\alpha_i^* = [\partial^2 u_i/\partial e^2]$ and Ω stands for mean curvature of $\Sigma(t)$ defined by $2\Omega = g^{\alpha\beta}b_{\alpha\beta}$. On the other hand, the second of eqns (2.5) gives

$$[\ddot{u}_i] = \alpha_i^* G_e^2 - 2G_e\frac{\delta\alpha_i}{\delta t} - \alpha_i\frac{\delta G_e}{\delta t}. \tag{3.17}$$

Now we begin deriving the equations of growth and decay for the transverse and longitudinal waves, separately.

(a) *Transverse wave*

We had previously found that the discontinuity vector is perpendicular to the propagation direction for transverse waves, i.e.

$$\alpha_k e_k = 0. \tag{3.18}$$

If we differentiate this relation with respect to u^α and use the relation [9]

$$e_{i,\alpha} = -g^{\beta\lambda}b_{\beta\alpha}x_{i\lambda}, \tag{3.19}$$

we get

$$\alpha_{k,\alpha}e_k = \alpha_k g^{\sigma\lambda}b_{\sigma\alpha}x_{k\lambda}. \tag{3.20}$$

Multiplying both sides of the second of eqns (3.16) by α_i, summing over i, and using eqns (3.15), (3.18) and (3.20), we obtain

$$[u_{k,ki}]\alpha_i = 0. \tag{3.21}$$

When we multiply eqn (3.11) by α_i and sum over i, and use eqns (3.14), $(3.16)_1$, (3.17), (3.18), (3.21) and the relation $G_e^2 = \mu(0)/\rho$, we obtain

$$\frac{\delta W}{\delta t} + \left\{\left(\frac{1}{2\rho G_e}\frac{\delta}{\delta t}(\rho G_e) + \frac{1}{G_e}\frac{\delta G_e}{\delta t}\right) - \Omega G_e + \frac{1}{2}m_s\right\}W = 0, \tag{3.22}$$

where W is the strength of the transverse wave defined by

$$W = (\alpha_i\alpha_i)^{1/2} \tag{3.23}$$

and

$$m_s = -\frac{\mu'(0)}{\mu(0)}. \tag{3.24}$$

(b) *Longitudinal wave*

For the longitudinal wave the discontinuity vector is parallel to the direction of propagation, i.e.

$$\alpha_k = \alpha e_k. \tag{3.25}$$

If we differentiate eqn (3.25) with respect to u^α and use the relation (3.19), we get

$$\alpha_{k,\alpha} = \alpha_{,\alpha} e_k - \alpha g^{\beta\lambda} b_{\beta\alpha} x_{k\lambda}. \tag{3.26}$$

Multiplying the second of eqn (3.16) by e_i and summing over i, and using eqns (3.15), (3.25) and (3.26), we find the relation

$$[u_{k,ki}]e_i = \alpha^*_k e_k - 2\Omega\alpha. \tag{3.27}$$

Equation (3.11), when multiplied by e_i and summed over i, yields

$$\frac{\delta W}{\delta t} + \left\{\left(\frac{1}{2\rho G_e}\frac{\delta}{\delta t}(\rho G_e) + \frac{1}{G_e}\frac{\delta G_e}{\delta t}\right) - \Omega G_e + \frac{1}{2}m_L\right\} W = 0, \tag{3.28}$$

in view of eqns $(3.9)_1$, (3.14), (3.16), (3.17), (3.25) and (3.27). In eqn (3.28), $W = \alpha$ describes the strength of the wave while

$$m_L = -\frac{\lambda'(0) + 2\mu'(0)}{\lambda(0) + 2\mu(0)} \tag{3.29}$$

and G_e is given by eqn $(3.9)_1$.

The equations of growth and decay for particle velocities and stresses for both the transverse and longitudinal waves can be obtained from eqns (3.22) and (3.28) by employing the constitutive relation (3.4), and eqns (3.6) which relate strains and particle velocities to the discontinuity vector. They are

$$\begin{aligned} &\frac{\delta\beta}{\delta t} + \left\{\frac{1}{2\rho G_e}\frac{\delta}{\delta t}(\rho G_e) - G_e\Omega + \frac{1}{2}m\right\}\beta = 0 \quad \text{for velocity discontinuity} \\ &\frac{\delta\gamma}{\delta t} + \left\{-\frac{1}{2\rho G_e}\frac{\delta}{\delta t}(\rho G_e) - G_e\Omega + \frac{1}{2}m\right\}\gamma = 0 \quad \text{for stress discontinuity,} \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} &\beta = [v_e]; \quad \gamma = [\sigma_{ee}]; \quad m = m_L; \quad G_e^2 = \frac{\lambda(0) + 2\mu(0)}{\rho} \quad \text{for longitudinal wave} \\ &\beta = [v_t]; \quad \gamma = [\sigma_{et}]; \quad m = m_s; \quad G_e^2 = \frac{\mu(0)}{\rho} \quad \text{for transverse waves.} \end{aligned} \tag{3.31}$$

In eqns (3.31), v_e and σ_{ee} denote the normal particle velocity and stress whereas v_t and σ_{et} represent the tangential particle velocity and stress at the wave front.

4. SOLUTIONS OF GROWTH AND DECAY EQUATIONS

The growth or decay of the particle velocity discontinuity β and the stress discontinuity γ can be found by integrating eqn (3.30). They are

$$\beta = \beta_0 \exp\left(-\int_0^t s_1(\tau)\,\mathrm{d}\tau\right); \quad \gamma = \gamma_0 \exp\left(-\int_0^t s_2(\tau)\,\mathrm{d}\tau\right), \tag{4.1}$$

where

$$s_1(t) = \frac{1}{2\rho G_e}\frac{\delta}{\delta t}(\rho G_e) - G_e\Omega + \frac{1}{2}m$$
$$s_2(t) = \frac{-1}{2\rho G_e}\frac{\delta}{\delta t}(\rho G_e) - G_e\Omega + \frac{1}{2}m, \quad (4.2)$$

and β_0 and γ_0 are the values of the velocity and stress discontinuities at the initial time $t = 0$, respectively.

In eqns (4.2), the first term on the right hand sides of the expressions for $s_1(t)$ or $s_2(t)$ describes the effect of nonhomogeneity, the second term represents that of the geometry of the wave front, while the last term corresponds to the influence of the material internal friction. We should note that the geometry of the wave front and therefore the mean curvature Ω, as well as the term m representing the effect of internal friction may be influenced by the nonhomogeneity. From the study of eqns (3.24) and (3.29) and the experimental observations, showing the negativeness of the time derivatives of the relaxation functions at $t = 0$, we further note that m should be positive, implying that the internal friction causes the strength of the wave to decay as the wave penetrates into the medium.

We now concentrate our attention on the evaluation of the integrals appearing in eqns (4.1). We first notice that the equations of growth and decay, eqns (4.1), are valid along the rays which carry the wave front as it propagates. Hence, to obtain the explicit forms of the growth and decay equations, the ray equations should first be found so that ρ, G_e, Ω, m and therefore s_1 and s_2 can be determined as functions of time along the rays. The canonical forms of the ray equations for the general anisotropic and nonlinear case are discussed in detail by Jeffrey and Taniuti[10]. In the present study where the medium is assumed to be linear, isotropic and nonhomogeneous, the ray equations take the forms

$$\frac{dx_i}{dt} = \frac{p_i G_e}{(p_k p_k)^{1/2}}$$
$$\frac{dp_i}{dt} = -(p_k p_k)^{1/2}\frac{\partial G_e}{\partial x_i}, \quad (4.3)$$

where $p_i = \partial\phi/\partial x_i$ and $\phi(x_i, t) = 0$ is the equation of the wave front. We note that **p** is normal to the wave front, therefore is parallel to the unit normal **e**. By solving the system of six first order differential equations, eqns (4.3), together with the proper initial conditions, the equations of rays $x_i = x_i(t)$, and $p_i = p_i(t)$ can be found.

5. PROPAGATION OF INITIALLY PLANE WAVES IN NONHOMOGENEOUS VISCOELASTIC MEDIA

In this section we study the propagation of initially plane waves for three types of inhomogeneities. In the first two cases, the inhomogeneity is assumed to be of axisymmetrical type whereas in the last case it is unidirectional.

(a) *Axisymmetrical inhomogeneity with radially increasing wave velocity*

Here, the medium is referred to a cylindrical coordinate system (r, θ, z) in which the $z = 0$ plane coincides with the initial plane wave front Σ_0 (Fig. 2). Further, we assume an inhomogeneity such that the wave propagation velocity depends on the radial coordinate r only, with the distribution

$$G_e = \begin{cases} G_e^0\left(1 + \dfrac{r}{a}\right) & \text{for } r < b \\ G_e^0\left(1 + \dfrac{b}{a}\right) & \text{for } r \geq b \end{cases}, \quad (5.1)$$

where a is a positive constant so that G_e increases linearly up till $r = b$ and remains constant thereafter. When, in view of the axisymmetrical nature of the problem, eqns (4.3) are integrated for this distribution of wave velocity together with the initial conditions

$$r = r_0; \quad z = 0, \quad \text{at } t = 0$$
$$p_r = 0; \quad p_z = 1, \quad \text{at } t = 0, \tag{5.2}$$

the ray equations in nondimensional forms are found to be

$$\bar{r} = \bar{r}_0; \quad \bar{z} = (1+\bar{b})\bar{t} \quad \text{for } \bar{r} > \bar{b} \tag{5.3}$$

$$\bar{r} = \frac{1+\bar{r}_0}{\cosh \bar{t}} - 1; \quad \bar{z} = (1+\bar{r}_0)\tanh \bar{t} \quad \text{for } \bar{r} < \bar{b}, \tag{5.4}$$

where the nondimensional distances and time are defined as

$$(\bar{r}, \bar{r}_0, \bar{z}, \bar{b}) = \frac{1}{a}(r, r_0, z, b); \quad \bar{t} = \frac{G_e{}^0 t}{a}. \tag{5.5}$$

Equations (5.3) show that the rays in the region $\bar{r} > \bar{b}$ are parallel straight lines that are perpendicular to the initial wave front Σ_0.

For $\bar{r} < \bar{b}$, however, the rays are not straight lines as seen from eqns (5.4). In fact, when the time $\bar{t}$ is eliminated between the first and the second of eqns (5.4), we find

$$\bar{z} = \{(1+\bar{r}_0)^2 - (1+\bar{r})^2\}^{1/2}, \tag{5.6}$$

indicating that the rays are circular arcs with the radius $(1+\bar{r}_0)$ and the center $(\bar{r}, \bar{z}) = (-1, 0)$ in the rz-plane. We note that eqns (5.4) or (5.6) are valid before the ray intersects the $\bar{z}$ − axis, i.e.

$$\bar{t} \le \bar{t}_{cr} = \cosh^{-1}(1+\bar{r}_0), \tag{5.7}$$

which follows from eqn (5.4), and the fact that $\bar{r} \ge 0$. In Fig. 2, drawn for $\bar{b} = 3$, the rays for various values of $\bar{r}_0$ are shown before $\bar{t}$ reaches $\bar{t}_{cr}$. To find the ray after it intersects the $\bar{z}$ − axis (i.e., $\bar{t} > \bar{t}_{cr}$), the ray equations, eqns (4.3), are again to be solved, but this time subject to the initial conditions which are found by imposing that the slope corresponding to the initial values of (p_r, p_z) is equal to that of the intersecting ray at the point of intersection. The analysis based on these arguments indicates that the rays for $\bar{r} < \bar{b}$ have periodic forms composed of circular arcs and with the periods equal to $4\sqrt{[(1+\bar{r}_0)^2 - 1]}$. For illustration the ray emanating from the point $\bar{r}_0 = 1.5$ is shown in Fig. 2.

The wave front at a fixed time $\bar{t} = \bar{t}^*$ can be found from the loci of the points on the rays corresponding to that time. The wave fronts, so obtained, are shown in Fig. 2 for the times $\bar{t} = 1.76$, 2.07 and 2.29. We here note that $\bar{t} = 2.07$, which is the time of arrival of the ray emanating from the point $\bar{r}_0 = \bar{b} = 3$ at the $\bar{z}$ − axis, can be considered as a critical time for the forms of the wave fronts, and that the times chosen are less than, equal to and greater than this value. We now study in detail the form of the wave front at $\bar{t} = 1.76$. It is composed of two parts which we call the leading and the lagging wave fronts. As shown in Fig. 2, the leading wave front consists of the conic surface FCF' and the planes DE and $D'E'$ while the lagging portion is a surface of revolution described by $AB'CB$. It should be noted that a small portion of the lagging wave front on the left of $AB'CB$, located between the points A and $\bar{z} = 1.76$, is omitted for simplicity. We also point out that the lagging wave front brings some additional disturbance to the region where the leading wave front has already passed. The study of Fig. 2 further shows that the region enclosed by $DGFCF'G'D'$ seems theoretically undisturbed due to the discontinuity of the derivative of $G_e(r)$ at $\bar{r} = \bar{b}$. However, if we smooth out the corner seen in the distribution of $G_e(r)$ at $\bar{r} = \bar{b}$ by a very small circular arc, the theoretical undisturbed region mentioned above will be filled out by the diverging secondary rays emanating from G and G' which form the secondary wave front described by the lines DF and $D'F'$. Both the secondary rays and the wave fronts are drawn approximately and are shown by dotted lines in Fig. 2. It should be noted that in the interval $0<t<2.07$, the apex angle of the conic part of the leading wave front decreases as the time increases. Further, the wave fronts corresponding to the times $\bar{t}=2.07, 2.29$ have similar forms as that obtained for $\bar{t}=1.76$; they, however, have no conical portions (Fig. 2).

The decay and growth equations along the circular rays before they intersect the $\bar{z}$ – axis will now be found by assuming that $\rho=\rho_0=$ constant and employing the general expressions, eqn (4.1). In this case the wave front is a conical surface having the mean curvature

$$\Omega=\frac{1}{2\bar{r}}. \tag{5.8}$$

When the wave velocity distribution, eqn (5.1), and the mean curvature expression, eqn (5.8), together with the first of eqns (5.4), describing the dependence of r on the time t, are used in eqns (4.2) and the integrations indicated in eqns (4.1) are performed, we find

$$\begin{aligned}\beta&=\beta_0\exp\left(-\frac{1}{2}\int_0^{\bar{t}}\bar{m}\,d\tau\right)(\cosh\bar{t})^{1/2}\left\{\frac{A\sqrt{(1-A^2)}\sinh\bar{t}+1-A^2}{A-A^2\cosh\bar{t}}-\frac{1}{A}\right\}^{1/[2\sqrt{(1-A^2)}]}\\ \gamma&=\gamma_0\exp\left(-\frac{1}{2}\int_0^{\bar{t}}\bar{m}\,d\tau\right)(\cosh\bar{t})^{-1/2}\left\{\frac{A\sqrt{(1-A^2)}\sinh\bar{t}+1-A^2}{A-A^2\cosh\bar{t}}-\frac{1}{A}\right\}^{1/[2\sqrt{(1-A^2)}]}\end{aligned} \tag{5.9}$$

which is valid for $\bar{t}\le\bar{t}_{cr}=\cosh^{-1}(1+\bar{r}_0)$ and where

$$A=\frac{1}{1+\bar{r}_0};\quad \bar{m}=\frac{a}{G_e^0}m. \tag{5.10}$$

The first factor in eqns (5.9) describes the internal friction effect on the decay of the velocity and stress discontinuities β and γ, discussed previously. The second factor in these equations represents the direct influence of the inhomogeneity which reveals as a growth on the velocity discontinuity and as a decay on the stress discontinuity as time increases, which is a natural consequence of decreasing wave velocity with increasing distance measured along the rays. The last factor denotes the geometrical effect due to material inhomogeneity and it is the same in both the velocity and the stress expressions. Since $\bar{r}_0>0$, the definition of A implies that $A<1$ and therefore $(1-A^2)>0$. When the geometrical factor is studied in view of this inequality, we conclude that this factor, starting with the value of 1 at $\bar{t}=0$, increases with time and becomes infinite for $\bar{t}=\bar{t}_{cr}$, at which the ray crosses the $\bar{z}$ – axis and thus the curvature Ω tends to infinity at the point of intersection. This, in turn, shows that the $\bar{z}$ – axis is a singular straight line along which the velocity and stress discontinuities increase beyond limit.

For the region $\bar{r}>\bar{b}$, the material is homogeneous and the rays are parallel straight lines (Fig. 2) forming plane wave fronts. In view of eqns (4.1), the particle velocity and the stress discontinuities suffer decays only due to material internal friction in this region.

(b) *Axisymmetrical inhomogeneity with radially decreasing wave velocity*

In this case the medium is again referred to a cylindrical coordinate system described in the previous problem. The distribution for the wave propagation velocity is assumed to be in the same form as in eqn (5.1), but this time a is a negative constant so that the wave velocity G_e decreases linearly up till $r=b$ and remains constant thereafter. Here b is assumed to be positive and smaller than $-a$ so that $G_e>0$ for all r.

The equations of rays emanating from the points of Σ_0 lying in the region $0<\bar{r}<\bar{b}$ can be found by employing a procedure similar to that used in the previous section. They, in nondimensional forms, are

$$\begin{aligned}\bar{r}&=1-\frac{1-\bar{r}_0}{\cosh\bar{t}}\\ \bar{z}&=(1-\bar{r}_0)\tanh\bar{t}\end{aligned} \tag{5.11}$$

and

$$\bar{z} = \{(1-\bar{r}_0)^2 - (1-\bar{r})^2\}^{1/2}, \tag{5.12}$$

where

$$(\bar{r}, \bar{r}_0, \bar{z}) = \frac{1}{-a}(r, r_0, z); \qquad \bar{t} = \frac{G_e^0}{-a} t. \tag{5.13}$$

The ray equations, eqns (5.11) and (5.12), are valid before the rays intersect the $\bar{r} = \bar{b} = (b/-a)$ cylindrical surface, i.e. in the time interval

$$\bar{t} \leq \bar{t}_{cr} = \cosh^{-1}\frac{1-\bar{r}_0}{1-\bar{b}}, \tag{5.14}$$

and they are circular arcs of radius $(1-\bar{r}_0)$ with the center at $(\bar{r}, \bar{z}) = (1, 0)$ as implied by eqn (5.12), and are shown in Fig. 3 plotted for $\bar{b} = 0.4$. for $\bar{t} > \bar{t}_{cr}$, the rays are straight lines which are tangent to the circular rays at the points of intersection with the surface $\bar{r} = \bar{b}$. The rays emanating from $\bar{r}_0 > \bar{b}$ are of course parallel straight lines perpendicular to the initial wave front Σ_0. The wave fronts are shown in Fig. 3 for three different times $\bar{t} = 0.96, 1.10$ and 1.50, where $\bar{t} = 1.10$, being the time of arrival of the ray emanating from $\bar{r}_0 = 0$ at $\bar{r} = \bar{b}$, is considered as a critical value. The wave fronts at these three different times have similar forms composed of leading and lagging parts; but the wave front for $\bar{t} < 1.10$ has a leading conical portion AB, whereas in the ones for $\bar{t} \geq 1.10$ this conical portion disappears (Fig. 3). The secondary wave fronts form due to the similar reasons discussed in the previous section and are shown in Fig. 3 by dotted lines.

The decay and growth equations abong the circular rays before they intersect the $\bar{r} = \bar{b}$ surface are found by assuming that the mass density is constant and by employing the general expressions, eqns (4.1), and the relation

$$\Omega = -\frac{1}{2\bar{r}} \tag{5.15}$$

for the mean curvature. They are

$$\beta = \beta_0 \exp\left\{-\frac{1}{2}\int_0^{\bar{t}} \bar{m}\, d\tau\right\}(\cosh \bar{t})^{1/2} \exp\left\{-\frac{1-\bar{r}_0}{\sqrt{[\bar{r}_0(2-\bar{r}_0)]}}\tan^{-1}\left\{\sqrt{\left(\frac{2-\bar{r}_0}{\bar{r}_0}\right)}\left(\frac{-1+\cosh \bar{t}}{\sinh \bar{T}}\right)\right\}\right\}$$

$$\gamma = \gamma_0 \exp\left\{-\frac{1}{2}\int_0^{\bar{t}} \bar{m}\, d\tau\right\}(\cosh \bar{t})^{-1/2} \exp\left\{-\frac{1-\bar{r}_0}{\sqrt{[\bar{r}_0(2-\bar{r}_0)]}}\tan^{-1}\left\{\sqrt{\left(\frac{2-\bar{r}_0}{\bar{r}_0}\right)}\left(\frac{-1+\cosh \bar{t}}{\sinh \bar{t}}\right)\right\}\right\} \tag{5.16}$$

which is valid for $\bar{t} \leq \bar{t}_{cr} = \cosh^{-1}\{(1-\bar{r}_0)/(1-\bar{b})\}$ and where

$$\bar{m} = \frac{-a}{G_e^0} m. \tag{5.17}$$

The effects of internal friction and direct material inhomogeneity on the growth and decay of velocity and stress discontinuities are the same as in the previous problem. However, the last factor in eqns (5.16), representing the geometrical effect due to material inhomogeneity, has a different form and will now be studied. Since $\bar{r}_0 \leq \bar{b} < 1$, the expression

$$\tan^{-1}\left\{\sqrt{\left(\frac{2-\bar{r}_0}{\bar{r}_0}\right)}\left(\frac{-1+\cosh \bar{t}}{\sinh \bar{t}}\right)\right\}$$

is zero at $\bar{t} = 0$ and then it continuously increases with time $\bar{t}$, which causes a decay in both the velocity and stress discontinuities as seen from eqns (5.16). The inspection of Fig. 3 further shows that the rays emanating from the region $0 \leq \bar{r}_0 \leq \bar{b}$ are parallel in the domain bounded by the surface $\bar{r}=\bar{b}$ whereas they, outside this domain, converge towards each other, which implies possible formation of caustics where high stress and velocity intensifications may occur (see Ref. [6] for more information about caustics).

(c) *Unidirectional inhomogeneity*

For this problem, the medium is referred to a cartesian coordinate system (x_1, x_2, x_3) where the $x_1=0$ plane is taken as the initial plane wave front Σ_0 (Fig. 4). It is assumed that the inhomogeneity occurs in the x_2-direction only, resulting a wave velocity distribution

$$G_e=\begin{cases} G_e^0\left(1+\dfrac{x_2}{a}\right) & \text{for } |x_2|\leq b \\ G_e^0\left(1+\dfrac{b}{a}\right) & \text{for } x_2>b \\ G_e^0\left(1-\dfrac{b}{a}\right) & \text{for } x_2<-b, \end{cases} \tag{5.18}$$

where a is a positive constant so that G_e, starting with the constant value of $G_e^0(1-b/a)$ in $x_2<-b$, increases linearly up to $x_2=b$ and remains constant thereafter.

The rays initiating from the region $|x_2|\leq b$ of the initial wave front have, in nondimensional forms, the equations

$$\bar{x}_2=\frac{1+\bar{x}_2^0}{\cosh \bar{t}}-1 \qquad \bar{x}_1=(1+\bar{x}_2^0)\tanh \bar{t} \tag{5.19}$$

and

$$\bar{x}_1=\{(1+\bar{x}_2^0)^2-(1+\bar{x}_2)^2\}^{1/2}, \tag{5.20}$$

which are obtained by the aid of the ray equations, eqns (4.3). The nondimensional quantities appearing in eqns (5.19) and (5.20) are defined as

$$(\bar{x}_1, \bar{x}_2, \bar{x}_2^0)=\frac{1}{a}(x_1, x_2, x_2^0); \qquad \bar{t}=\frac{G_e^0}{a}t, \tag{5.21}$$

and x_2^0 designates the initial coordinate of the ray. The ray equations, (5.19) and (5.20), are valid before the rays reach the plane $\bar{x}_2=-\bar{b}=-(b/a)$, i.e. in the time interval

$$\bar{t}\leq \bar{t}_{cr}=\cosh^{-1}\frac{1+\bar{x}_2^0}{1-\bar{b}}, \tag{5.22}$$

and they are circular arcs of radius $(1+\bar{x}_2^0)$ and of the center at $(\bar{x}_1, \bar{x}_2)=(0,-1)$ as seen from eqn (5.20), and are shown in Fig. 4 for $\bar{b}=0.4$. For $\bar{t}>\bar{t}_{cr}$, they become straight lines tangent to the circular rays at points of the plane $\bar{x}_2=-\bar{b}$. The rays, originating from the region $|\bar{x}_2^0|>\bar{b}$ of the initial wave front, are parrallel straight lines normal to Σ_0. The wave fronts, obtained with the aid of the rays, are shown in Fig. 4 for two different times $\bar{t}=1.21$ and 1.60. The secondary wave fronts are again shown by dotted lines in the same figure. Since the part AB of the leading wave front is plane, i.e. $\Omega=0$, the growth and decay equations along the circular rays, before they reach the $\bar{x}_2=-\bar{b}$ plane, become

$$\beta=\beta_0\exp\left\{-\frac{1}{2}\int_0^{\bar{t}}\bar{m}\,d\tau\right\}(\cosh \bar{t})^{1/2} \qquad \gamma=\gamma_0\exp\left\{-\frac{1}{2}\int_0^{\bar{t}}\bar{m}\,d\tau\right\}(\cosh \bar{t})^{-1/2} \quad \text{for } \bar{t}\leq \bar{t}_{cr}, \tag{5.23}$$

where $\bar{m} = (a/G_e^0)m$ and the mass density ρ is assumed to be constant. We note that the geometrical decay or growth due to the nonhomogeneity disappears in this case and that the converging rays reveal the formation of caustics in the region $\bar{x}_2 < -\bar{b}$.

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